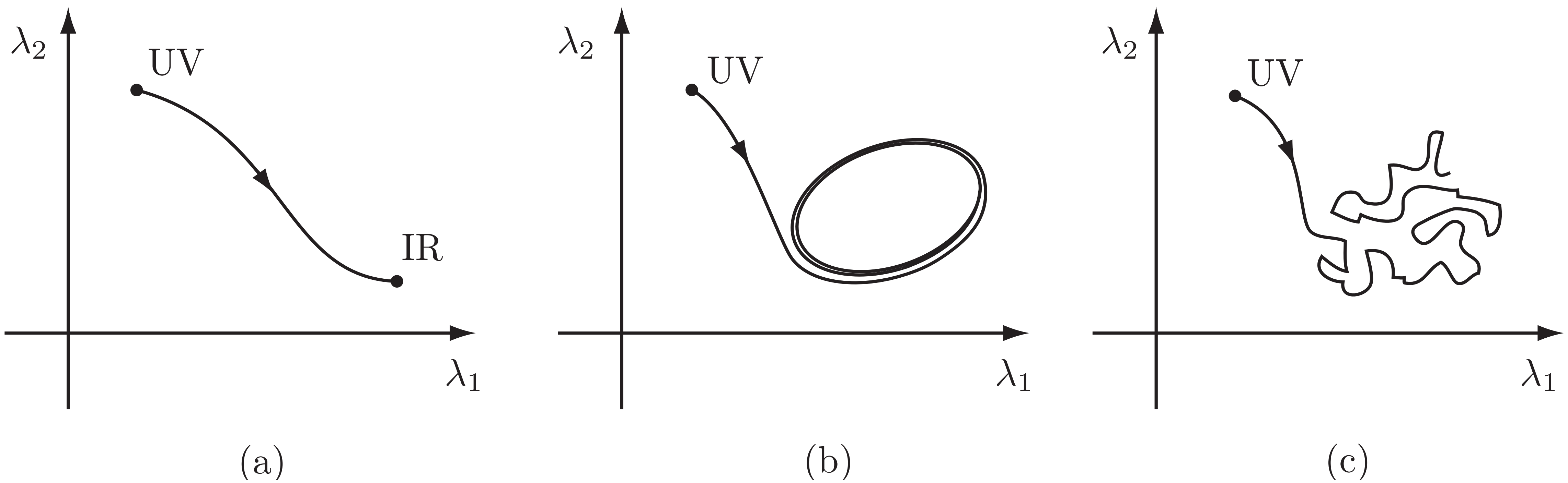
The a-theorem
and
the asymptotics of 4D-QFT

Riccardo Rattazzi

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

M. Luty, J. Polchinski, RR arXiv:1204.5221

conceivable RG flows



but all known examples asymptote to CFT

- free (QED, massless QCD)
- strongly coupled (Supersymmetry)
- trivial (real QCD)

a-theorem suggests constraints on other options

✦ Learning about QFT by plugging it in external metric (dilaton background)

✦ Ruling out non-CFT asymptotics in perturbation theory

✦ Towards a non-perturbative proof

QFT in gravity background

$$\hat{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}$$

dilaton background

$$g_{\mu\nu} = \eta_{\mu\nu}$$

$$\Omega \equiv 1 + \varphi$$

quantum effective action $W[\hat{g}_{\mu\nu}]$

$$\frac{1}{\sqrt{g}} \frac{\delta}{\delta \hat{g}^{\mu\nu}(x)} \equiv T_{\mu\nu}(x)$$

- diff invariant
- finite up to local counterterms

possible counterterms $(\,\mathrm{dim}\ \leq 4\,)$

$\sqrt{\hat{g}}$	$\sqrt{\hat{g}}R$	$\sqrt{\hat{g}}R^2$	$\sqrt{\hat{g}}E_4$	$\sqrt{\hat{g}}W^2$
↓	↓	↓	↓	↓
Ω^4	$\Omega\Box\Omega$	$\Omega^{-2}(\Box\Omega)^2$	0	0

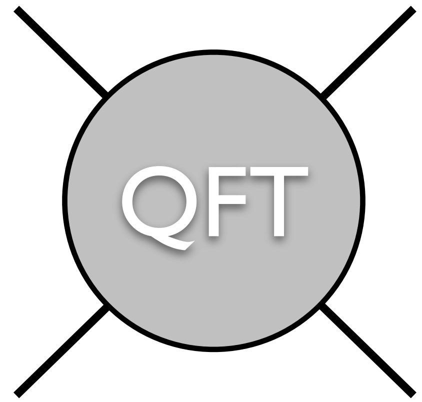
On shell dilaton : $\Box\Omega = 0$

On-shell dilaton amplitudes are fixed by flat limit QFT

modulo C.C. term

$$A(p_1,\ldots,p_4) = \frac{\delta^4 W}{\delta\varphi(p_1)\cdots\delta\varphi(p_4)} = \langle T(p_1)T(p_2)T(p_3)T(p_4)\rangle + \text{contact terms.}$$

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$$



$= A(s,t)$

In CFT $A(s,t)$ finite and yet local

Weyl anomaly

$$W_{CFT}[\Omega^2 g_{\mu\nu}] = W_{CFT}[g_{\mu\nu}] - S_{\rm WZ}[g_{\mu\nu}, \Omega; a, c]$$

Tomboulis 1990
Schwimmer, Theisen '11

$$-2\,a\,(\partial \ln \Omega)^4$$

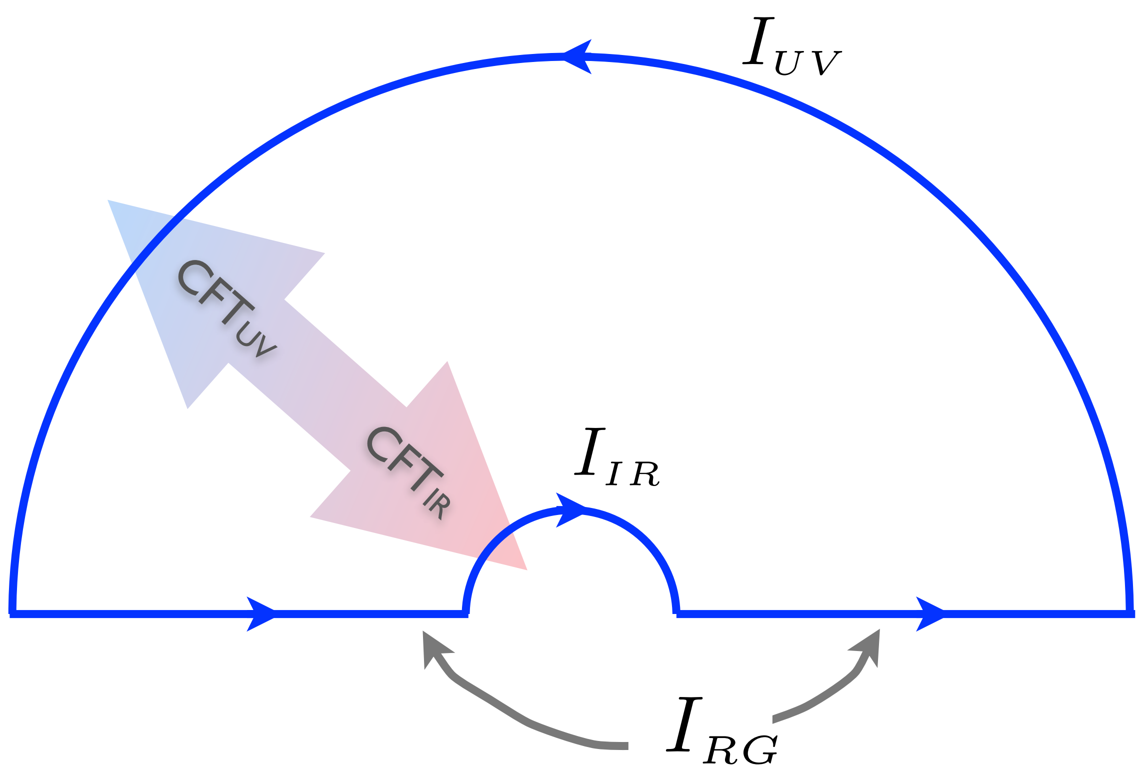
$$A(s,t) \longleftrightarrow a$$

Komargodski, Schwimmer '11

In approximate CFT a = running coupling in (Wilsonian) effective action

CFT_{UV}

CFT_{IR}



$$\oint_C \frac{A(s,0)}{s^3} ds = 0$$

$$I_{IR} + I_{UV} + I_{RG} = 0$$

$$a_{IR} = a_{UV} - I_{RG} \quad < a_{UV}$$

$$I_{RG} = \frac{1}{4\pi}\int \frac{\operatorname{Im} A}{s^3} = \frac{1}{4\pi}\int \frac{\sigma}{s^2} > 0$$

- I_{RG} is nicely finite in CFT-to-CFT flows
- It had to be so, cause A'' does not require renormalization; it is just a function of the renormalized QFT couplings
- Finiteness of I_{RG} → constraint on QFT asymptotics

✦ Learning about QFT by plugging it in external metric $\left(\begin{array}{c} \text{dilaton} \\ \text{background} \end{array} \right)$

✦ Ruling out non-CFT asymptotics in perturbation theory

$$\text{study} \quad \frac{da}{d\ln\Lambda} = \frac{dI_{RG}}{d\ln\Lambda}$$

✦ Towards a non-perturbative proof

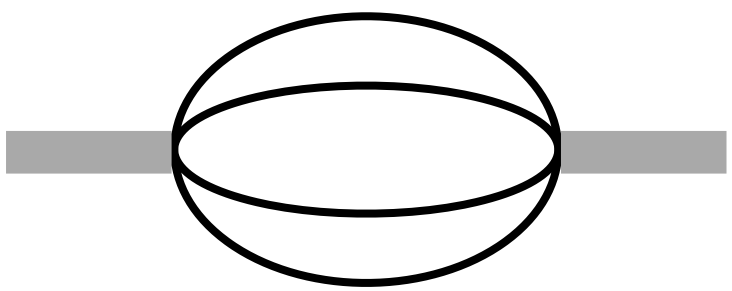
$$S[\Omega^2\eta_{\mu\nu},\, \hat{\Phi},\, m_i] \;=\; S[\eta_{\mu\nu},\, \Phi,\, \Omega m_i] \qquad\qquad \Phi \;=\; \Omega^{\Delta_\Phi}\hat{\Phi}$$

marginal coupling in dim reg

$$\lambda_0 \;=\; \mu^\epsilon P(\lambda(\mu),\epsilon) \quad \Rightarrow \quad \Omega^\epsilon\mu^\epsilon P(\lambda(\mu),\epsilon) \;\overset{\text{RG inv}}{=}\; \mu^\epsilon P(\lambda(\Omega\mu),\epsilon)$$

$$\mathcal{L}_{int} \;=\; \beta_\lambda \,\ln\Omega\,\frac{\Phi^4}{4!}\left(1 + \frac{(\ln\Omega)}{2}\,\partial_\lambda\beta_\lambda + \ldots\right)$$

$$\Delta W^{leading} \;=$$



$$\langle 0|\mathrm{T}\,\Phi^4(x)\Phi^4(y)|0\rangle$$

Unitarity

$$c_\lambda = \frac{1}{2^{12}(4!)^2\pi^6} \qquad \Delta W^{leading} \;=\; +\,c_\lambda\,\beta_\lambda^2\,\ln\mu^2\int d^4x\,(\partial\ln\Omega)^4.$$

$$\mu\frac{da}{d\mu} = c_\lambda\beta_\lambda^2 \;>\; 0$$

Analogous computation for gauge and Yukawa couplings

$$\mu\frac{da}{d\mu} = \sum_A c_A \beta_A^2 \qquad\qquad c_A > 0$$

$$a_{UV} - a_{IR} = \int_{t_{IR}}^{t_{UV}} dt \sum_A c_A \beta_A^2 \quad (1 + O(\alpha))$$

Jack, Osborn 1984

integral must converge : two cases

I. theory exits perturbative regime : can't say much in general

II. throughout RG flow $O(\alpha) \ll 1$ $\qquad \int dt \beta^2 < \infty$

$$\lim_{t\to\pm\infty} \beta_A(t) = 0$$

a-finiteness → weakly coupled asymptotics must be free CFTs

$$\beta_A = 0$$

Corollary: perturbative SFT asymptotics are ruled out

SFT $\quad \beta_A \neq 0 \quad \sum_A \beta_A^2 = C = \text{const}$

Fortin, Grinstein, Stergiou '11

$$a_{UV} - a_{IR} = C \ln \frac{\Lambda_{UV}}{\Lambda_{IR}}$$

but no counterterm exists to account for this log

⇓

$C = 0 \qquad \beta_A = 0$

- do not see how to carry out argument to SFTs in $D = 4 - \epsilon$
- E_4 non vanishing in $4 - \epsilon$
- 'quick analysis' shows no contradiction as long as $|\beta_A| < \sqrt{\epsilon}$

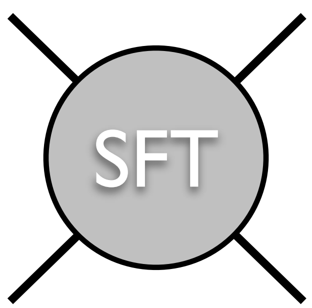
Non perturbative argument contra 4D SFTs

$$S^\mu = T^\mu{}_\nu \, x^\nu + V^\mu \qquad\qquad 0 = \partial_\mu S^\mu \;=\; T^\mu{}_\mu + \partial_\mu V^\mu$$

SFT $$T \;\equiv\; T^\mu{}_\mu \;=\; -\partial_\mu V^\mu \;\neq\; 0$$

Wess 1960
Polchinski 1988

$A(s,t) \;=$



$$= \langle T(p_1)T(p_2)T(p_3)T(p_4)\rangle \;+\; \langle T(p_1+p_2)T(p_3)T(p_4)\rangle + \text{permutations}$$
$$+\,\langle T(p_1+p_2)T(p_3+p_4)\rangle + \text{permutations}$$
$$+\,\langle T(p_1+p_2+p_3)T(p_4)\rangle + \text{permutations}$$

$T \;\neq\; 0$

In SFT one would expect amplitude to be non-local

and, in particular, $\mathrm{Im}\, A(s,0) \;\neq\; 0$

$\text{Im}\, A(s,0)$ is constrained by scale invariance and unitarity

$$\text{Im}\, A(s,0) \;=\; C\, s^2 \qquad\qquad C \;\geq\; 0$$

dispersion relation

$$a_{IR} \;-\; a_{UV} \;=\; -\frac{1}{4\pi}\int \frac{\text{Im}\, A}{s^3} \;=\; \frac{C}{4\pi}\,\ln\frac{\Lambda_{IR}}{\Lambda_{UV}}$$

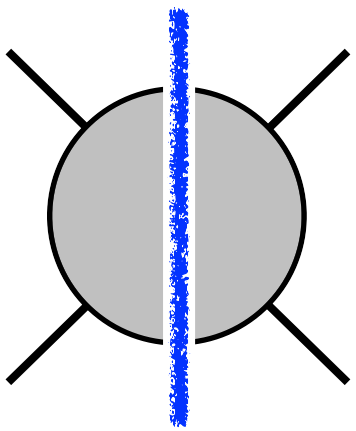
absence of candidate counterterm ⇒

$$C \;=\; 0$$

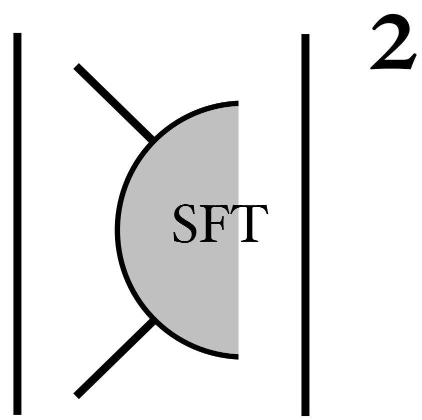
$$\text{Im}\, A(s,0) \;=\; 0$$

optical theorem

$$\mathrm{Im}\, A(s,0) \;=\; s\,\sigma(\varphi\varphi \to \mathrm{SFT}) \;\times\; f^4$$



=



$$= \sum_{\Psi} \left| \langle \Psi |\, \mathrm{T}\{T(p_1)T(p_2)\} + T(p_1+p_2)\, |0\rangle \right|^2$$

$$\mathrm{Im}\, A(s,0) \;=\; 0 \quad \xleftrightarrow{\text{unitarity}} \quad \mathrm{T}\{T(p_1)T(p_2)\} + T(p_1+p_2) \;=\; 0$$

$$\mathrm{T}\{T(p_1)T(p_2)\} + T(p_1+p_2) \; = \; 0$$

- Near conformal case (ex. perturbative case)

$$\mathrm{T}\{T(p_1)T(p_2)\} \sim \beta^2 \;\ll\; \beta \sim T(p_1+p_2)$$

constraint satisfied only for $T(p_1+p_2) \; = \; 0$ ➡ CFT

all boils down to $\langle \Psi | T(p_1+p_2) | 0 \rangle \; = \; 0$ like in 2D proof

- p_1 et p_2 are not arbitrary $p_1^2 \; = \; p_2^2 \; = \; 0$

cannot yet directly infer $\mathrm{T}\{T(x_1)T(x_2)\} + \delta^4(x_1 - x_2)\, T(x_1) \; = \; 0$

and conclude T is trivial

The importance of Unitarity

- Non-unitary SFT: $T \neq 0$ can be compatible with $\mathrm{Im}\, A = 0$ thanks to cancellation between positive and negative norm state

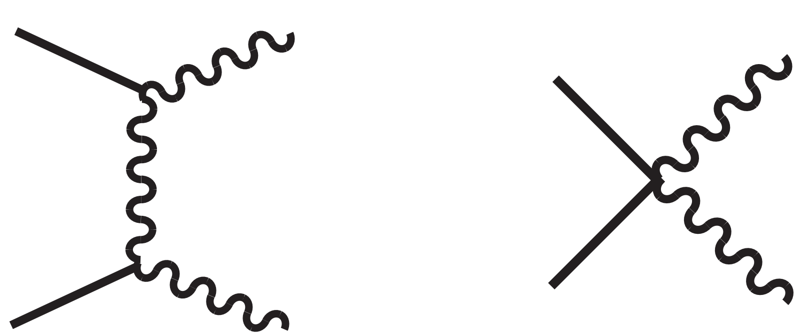
- no log divergence in a : **must** have $\mathrm{Im}\, A = 0$

- check: massless vector without gauge invariance

$$S = \int d^4x \sqrt{-\hat{g}} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{h}{2} (\nabla_\mu A^\mu)^2 \right)$$

Coleman, Jackiw 1971
Riva, Cardy 2005

SFT with virial current $V^\mu = h\, A_\nu F^{\mu\nu}$



$\neq 0$ for exclusive final states

$= 0$ inclusive

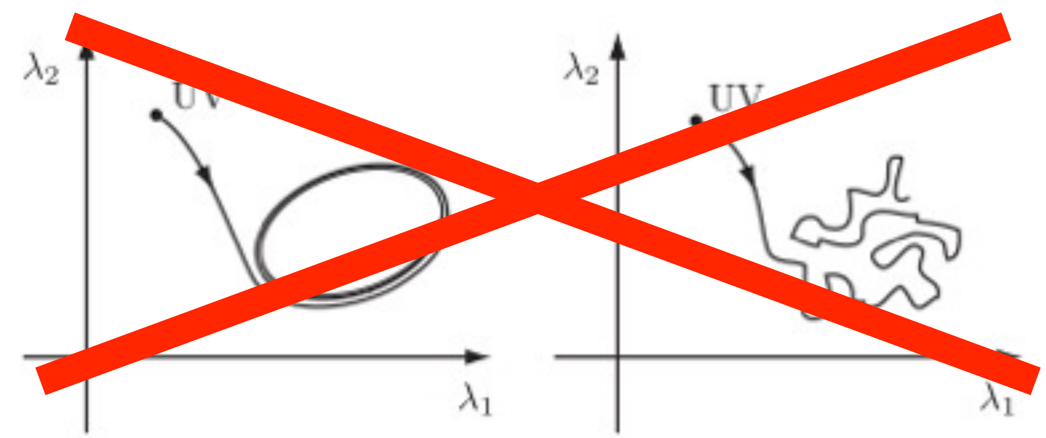
Summary

Finiteness of total RG flow of a-anomaly

Monotonicity $\mu \frac{da}{d\mu} \geq 0$

Powerful constraint on RG-flow

✦ Perturbative theories $\lim_{\ln \mu \to \pm\infty} \beta = 0$



predict FGS 4D examples will turn out to be CFTs and not SFTs

✦ Same conclusion for *small* deformations of strongly coupled CFTs

same conclusion, different method, in supersymmetry Antoniadis, Buican '11

✦ Non-perturbative constraint on SFTs

$$\langle\Psi|\, \mathrm{T}\big\{T(p_1)T(p_2)\big\} + T(p_1+p_2)\,|0\rangle \;=\; 0 \qquad\qquad \forall\,\Psi$$

very close to implying $\quad T \;\equiv\; T^{\mu}_{\mu} \;=\; 0 \quad$ but not there yet

✦ Non-unitary example (theory of elasticity in 4D classical stat mech)

$$\langle\Psi|\, \mathrm{T}\big\{T(p_1)T(p_2)\big\} + T(p_1+p_2)\,|0\rangle \;\neq\; 0$$

while still $\quad \mu\dfrac{da}{d\mu} \;=\; 0 \quad$ ...as it must !