Scattering Amplitudes: from QCD to Strings

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We observe Nature by detecting photons (and more) scattered all over the Universe

Science is based on such observations

For the scientific understanding to be possible, the scattering amplitudes must contain complete information about physical processes

At the LHC, millions of protons collide every second, probing Nature at shortest distances possible. Experimental data are analyzed and compared with theoretical computations of the scattering amplitudes involving electrons, photons, gluons, Higgs scalar etc.

Fortunately, elementary particle physicists had 50 years to prepare for the LHC. At this point, **almost** all scattering amplitudes necessary for studying the standard model and all conceivable BSM extensions are known, either in short analytical or in numerical form

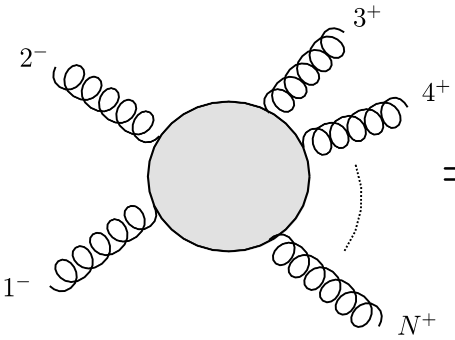
So why are **we** still excited about scattering amplitudes?

Over the last decade, many (famous) theorists became increasingly frustrated by tremendous amount of work necessary to compute some simple amplitudes

Take this one...

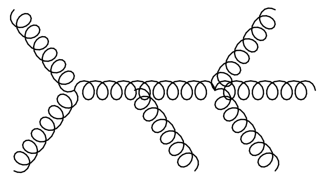
N-gluon maximally helicity violating (MHV) tree amplitude

in pure YM, QCD, SUSY QCD, you name it:



$$= g^{(N-2)} \mathrm{Tr}(T^1 T^2 \cdots T^N) \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle N{-}1 N \rangle \langle N1 \rangle} + \ldots$$

=



+... (Parke, T; Mangano, Parke, Xu; Berends, Giele, 1980s)

Notation:

Gamma matrices $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ $\quad \sigma = (\mathbf{1}, \vec{\sigma})$, $\bar{\sigma} = (\mathbf{1}, -\vec{\sigma})$ $\quad \gamma^5 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

Momenta p_i, $\;p^2 = \det(p_\mu \bar{\sigma}^\mu) = 0$

$$\Downarrow$$

$$(p_{i\mu} \bar{\sigma}^\mu)_{\alpha\dot{\alpha}} = \lambda_{i\alpha} \tilde{\lambda}_{i\dot{\alpha}} \qquad \lambda_i \equiv u_-(p_i) \qquad \tilde{\lambda}_i \equiv \bar{u}_-(p_i)$$

$$\langle ij \rangle = \epsilon^{\alpha\beta} \lambda_{i\alpha} \lambda_{j\beta} \qquad [ij] = -\epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_{j\dot{\beta}}$$

This is not the only one... Other (NMHV) tree amplitudes depend on both λ and $\tilde{\lambda}$, but are still very simple. Even the loop corrections, especially in N=4 SYM are not so complicated. Instead of blaming Feynman, we should ask:

What are we missing? Usually, simplicity hints at symmetries. We have Poincare, gauge, scale invariance (at tree level) and ?

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What are we missing? Usually, simplicity hints at symmetries. We have Poincare, gauge, scale invariance (at tree level) and actually, we have full

Conformal Symmetry

$P^\mu, M_{\mu\nu}, D$(ilatation) $\quad K_\mu$(special conformal transformations)

$$P(a^\mu) : x^\mu \to x^\mu + a^\mu$$

$$K(b^\mu) : x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2bx + b^2 x^2} = I e^{ibP} I(x^\mu)$$

where I is the inversion

$$I : x^\mu \to \frac{x^\mu}{x^2}$$

Conformal Group $SO(2,4) = SO(1+\overbrace{1,\ 3}^{M\ \text{(Lorentz)}}+1)$ ~ **SU(2,2)**

$$\underbrace{1+1,\ 3+1}_{K,P}$$

Fundamental representation:

$$\textbf{Twistors}\quad Z^I = (\lambda^\alpha, \mu^{\dot\alpha})$$ (Penrose, 1960s)

$$K, P \ : \quad \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \to \exp \begin{pmatrix} 0 & b_\mu \sigma^\mu \\ a_\mu \bar\sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$$

Amplitudes $(\lambda, \tilde\lambda)$ & Fourier-Penrose transform $\tilde\lambda \to \mu$

(Witten, 2003)

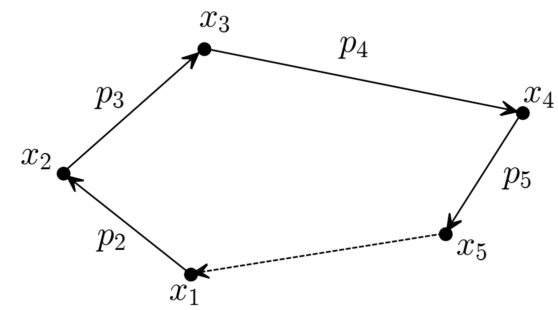
$$\Downarrow$$

Amplitudes (Z) in twistor space

explicit conformal invariance, elegant description in terms of holomorphic curves in $CP^{3|4}$ but no compelling reason for simplicity

Dual Conformal Symmetry : Coordinates

(Drummond, Henn, Korchemsky, Sokatchev, 2008)



$$x_i \;:\; p_i = x_i - x_{i-1} \overset{x_1=0}{\Longrightarrow} x_n = \sum_{k=2}^{n} p_k$$

$$\text{Inversion } x^\mu \to \frac{x^\mu}{x^2}$$

Momentum Twistors

(Hodges, 2009)

$$Z_i^I = (\lambda_i^\alpha \;,\; x_i^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \lambda_{i\alpha})$$

SU(2,2) invariants

$$\langle ijkl \rangle = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L$$

e.g. $$[2|3+4|5\rangle = \tilde{\lambda}_2 (p_3^\mu + p_4^\mu) \bar{\sigma}_\mu \lambda_5 = \frac{\langle 1235 \rangle}{\langle 12 \rangle \langle 23 \rangle}$$

All tree-level amplitudes covariant, written in terms of $\langle \ldots \rangle$

Loop Corrections

Conformal (and dual conformal) symmetry broken by regularization, but in a "controllable" way -- can be described as "anomalies"

Complexity increases with the loop number: integrals over Feynman parameters produce special functions of kinematic invariants: Spence's Li_2 is just a tip of an iceberg of polylogarithms etc. Some loop corrections have interesting "transcendental" properties which help in understanding their structure

1. *Philosophy*
a. Concerned with the a priori or intuitive basis of knowledge as independent of experience.
b. Asserting a fundamental irrationality or supernatural element in experience.
2. Surpassing all others; superior.
3. Beyond common thought or experience; mystical or supernatural.
4. *Mathematics* Of or relating to a real or complex number that is not the root of any polynomial that has positive degree and rational coefficients.

Later, I will discuss transcendentality of the low-energy expansions of superstring amplitudes – somewhat simpler because they arise from loop expansions of **two**-dimensional sigma models describing string world-sheets…

Superstring Amplitudes

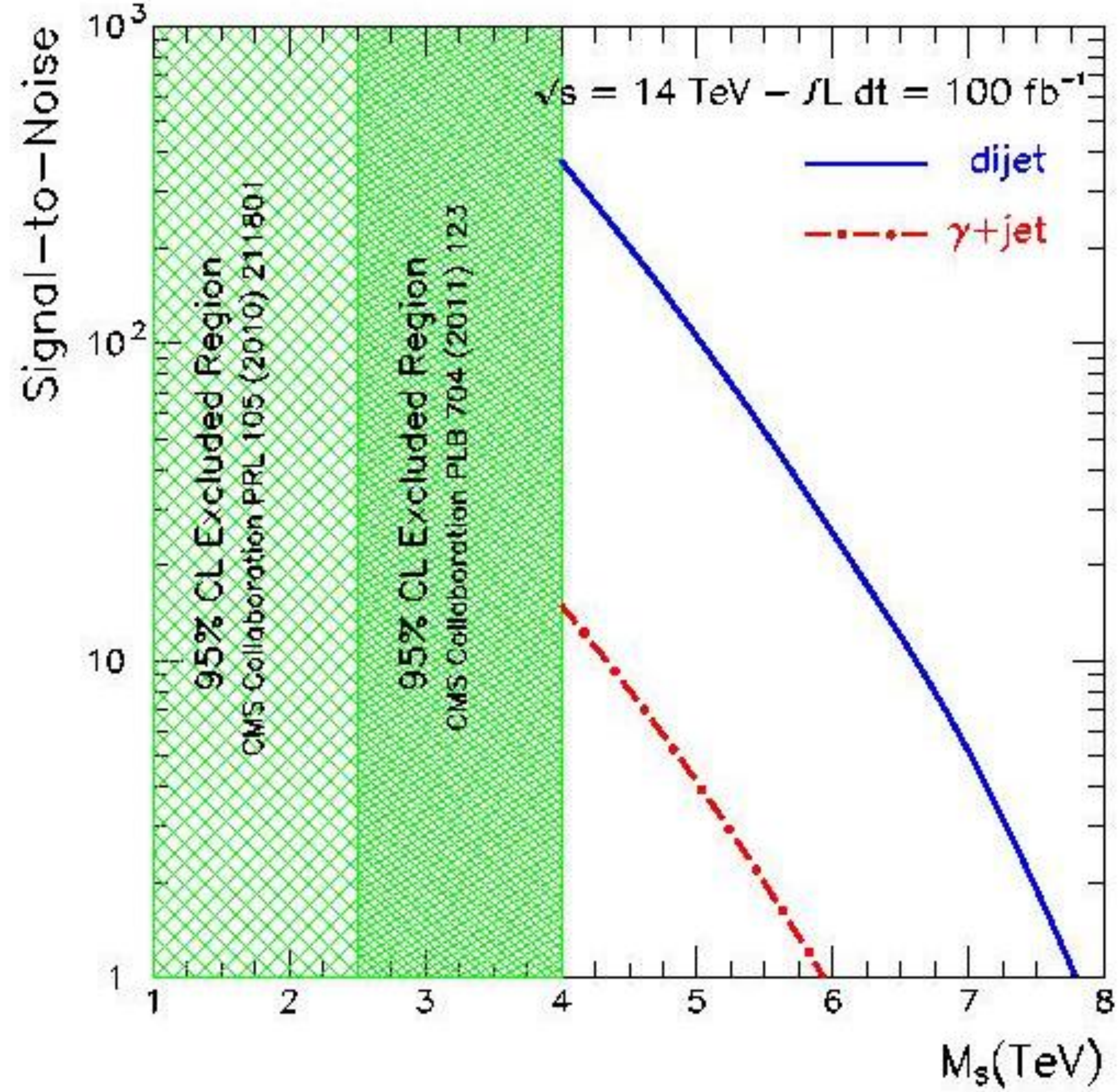
Superstring theory started in the late 1900's as a theory unifying gravity with the standard model. I will focus on open superstrings because they give rise to massless gauge bosons. They offer an interesting generalization (deformation) of gauge theories, characterized by just one parameter, the fundamental string mass M_s which violates scale (thus conformal) invariance.

Just a reminder - it is a theory of vibrating strings. Gluons appear among the zero modes. First harmonics give rise to massive gluons (mass M_s) and spins ranging from 0 to 2. The second harmonics have masses $\sqrt{2}M_s$ and spins up to 3... Gluons have infinite towers of massive, higher spin excitations (Regge resonances) populating "Regge trajectories" with the slope

$$\alpha' \sim \frac{1}{M_s^2} \;, \qquad \text{QCD in } \alpha' \to 0 \text{ low-energy limit}$$

Comment: M_s is an arbitrary parameter. If it were in few TeV-range, then many cross sections, like for the dijet production, would reveal heavy gluon resonances, hard to miss at the LHC...

(Antoniadis, Anchodroqui, et al, 2000s)

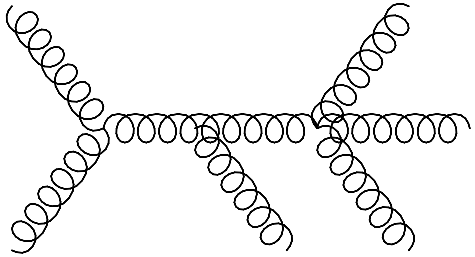


Signal−to−Noise
10³
10²
10
1
1
2
3
4
5
6
7
8
Ms(TeV)
√s = 14 TeV − ∫L dt = 100 fb⁻¹
dijet
γ+jet
95% CL Excluded Region
CMS Collaboration PRL 105 (2010) 211801
95% CL Excluded Region
CMS Collaboration PLB 704 (2011) 123

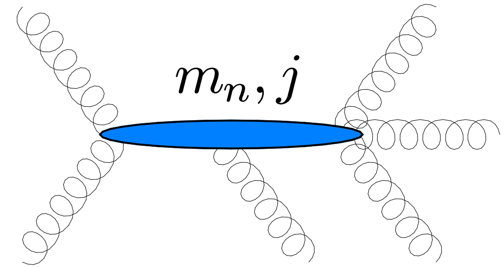
- Full-fledged superstring amplitudes are much harder to compute. Are they still “simple”? We will look at N-gluon MHV superstring amplitudes

- Are there any remnants of conformal and/or dual conformal symmetries in spite of explicit conformal symmetry breaking by the string mass?

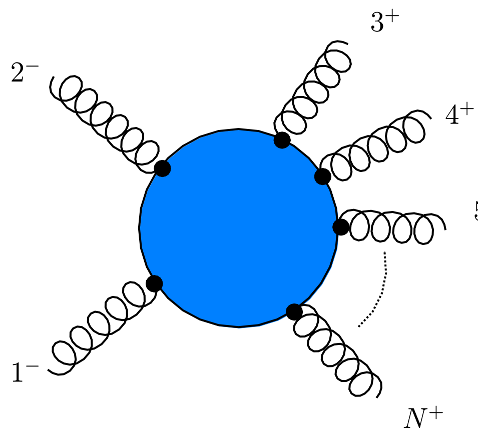
Now in the tree approximation, we need to include infinite number of Feynman diagrams with whole towers of resonances propagating in intermediate channels



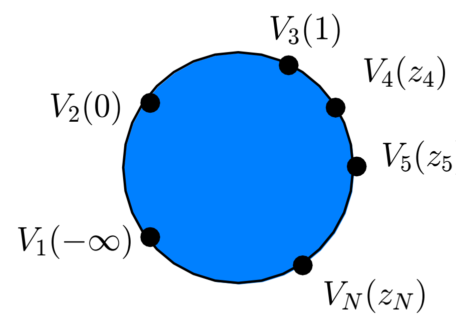
$$\rightarrow \sum_{m_n=\sqrt{n}M_s}$$



Fortunately, this can be done by computing just one **disk** diagram:



$$= \int \prod_{i=4}^{N} dz_i$$



$$= \ldots \left\langle V_1(-\infty)V_2(0)V_3(1)\int_{z_3}^{z_1} dz_4 V_4(z_4)\cdots\int_{z_{N-1}}^{z_1} dz_N V_N(z_N)\right\rangle$$

n.b. vertex positions similar to Feynman loop parameters

Transcendental Integrals

$$z_{ij} \equiv z_i - z_j$$

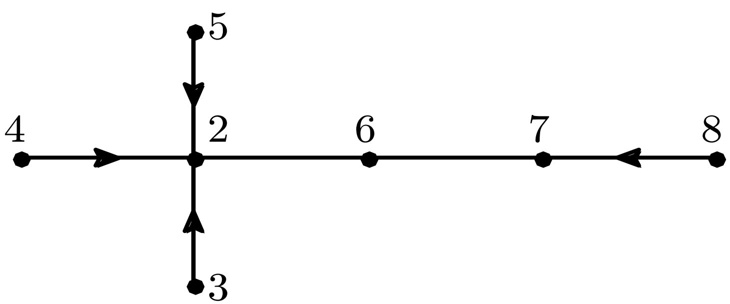
$$(ij) \equiv z_{ij}^{-1}$$

$$[\![(i_1 i_2)(i_3 i_4)\dots(i_{n-1} i_n)]\!]_N$$

$$= \int_1^\infty dz_4 \cdots \int_{z_{N-1}}^\infty dz_N \left(z_{i_1 i_2} z_{i_3 i_4} \cdots z_{i_{n-1} i_n}\right)^{-1} \prod_{2\le k<l\le N} (z_{kl})^{2\alpha' p_k p_l}$$

all $(z_{ij} z_{ji})^{-1} \dots, (z_{ij} z_{jk} \cdots z_{mi})^{-1}$ (i.e. closed cycles) cancel

$\Rightarrow$ can be represented by trees, e.g:



$$= [\![(42)(52)(32)(87)(76)(62)]\!]_8$$

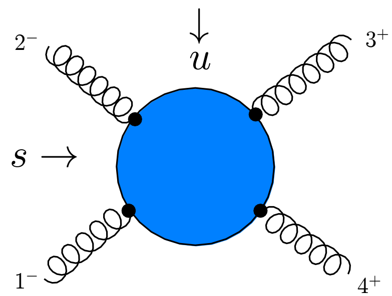
Trees="Transcendental" : **why?**

Transcendental Integrals

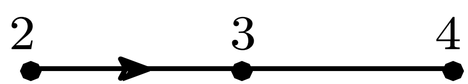
3. Beyond common thought or experience; mystical or supernatural.

4. *Mathematics* Of or relating to a real or complex number that is not the root of any polynomial that has positive degree and rational coefficients.

Example: N=4 (Veneziano-Virasoro) four-gluon amplitude



$$\hat{s} = 2\alpha' p_1 p_2, \quad \hat{u} = 2\alpha' p_2 p_3$$



$$= [\![(23)(34)]\!]_4 \overset{z_4 \to 1/x}{=} \int_0^1 dx\, x^{\hat{u}-1}(1-x)^{\hat{s}-1}$$

$$= B(\hat{u},\hat{s}) = \frac{1}{\hat{u}}\, {}_2F_1(\hat{u}, 1-\hat{s}; 1+\hat{u}, z=1)$$

QCD in $\alpha' = 0$ limit. $\alpha' \to 0 \quad (\hat{u},\hat{s} \ll 1)$ low-energy expansion:

$$[\![(23)(34)]\!]_4 \;=\; \underline{(\alpha')^{-1}}\left(\frac{1}{s}+\frac{1}{u}\right) - \underline{\alpha'\zeta(2)}(s+u) + \underline{(\alpha')^2\zeta(3)}(s+u)^2 + \ldots$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \qquad \zeta(2) = \frac{\pi^2}{6}$$

Degree of Transcendentality

(Kotikov, Lipatov , 2003)

$$DoT(r) = 0 \text{ for rational } r$$
$$DoT(\pi^k) = DoT(\zeta(k)) = k \text{ for positive } k$$
$$DoT(uv) = DoT(u) + DoT(v)$$

$$[\![(23)(34)]\!]_4 \;=\; \underline{(\alpha')^{-1}}(\tfrac{1}{s} + \tfrac{1}{u}) - \underline{\alpha'\zeta(2)}(s+u) + \underline{(\alpha')^2\zeta(3)}(s+u)^2 + \ldots$$

In general:

- $[\![\ldots]\!]_N \sim {}_pF_q(\ldots, z_1 = 1, \ldots)$ "multiple Gaussian hypergeometric etc."
- $\alpha' \to 0:$

$$[\![\ldots]\!]_N \sim \underline{(\alpha')^{3-N}}R_{3-N} + \underline{(\alpha')^{5-N}\zeta(2)}R_{5-N} + \underline{(\alpha')^{6-N}\zeta(3)}R_{6-N} + \ldots$$

$$\ldots + \underline{(\alpha')^{m-N}\mathcal{T}_{m-3}}R_{m-N} + \ldots$$

where $DoT(\mathcal{T}_n) = n$ and R_n are rational homogenous (degree n) functions of kinematic invariants $s_{ij} = 2p_ip_j$.

Basis of Integrals

$(N-3)!$ -element "chain" basis labeled by $\sigma \in \mathrm{Perm}(4,5,\ldots,N)$

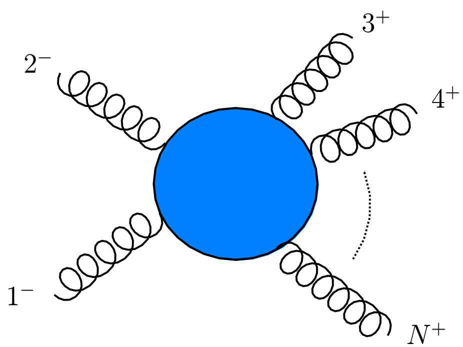
$$\mathcal{C}^N_\sigma \;=$$

$$= \;[\![(23)(34_\sigma)\ldots((N-1)_\sigma N_\sigma)]\!]_N$$

see also Stieberger, T, 2006

Mafra, Schlotterer, Stieberger, 2011

MHV Amplitude for Superstrings



$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\left(\prod_{k=4}^{N}\langle 2k\rangle\right)^{-1}\times$$

$$\times \sum_\sigma$$

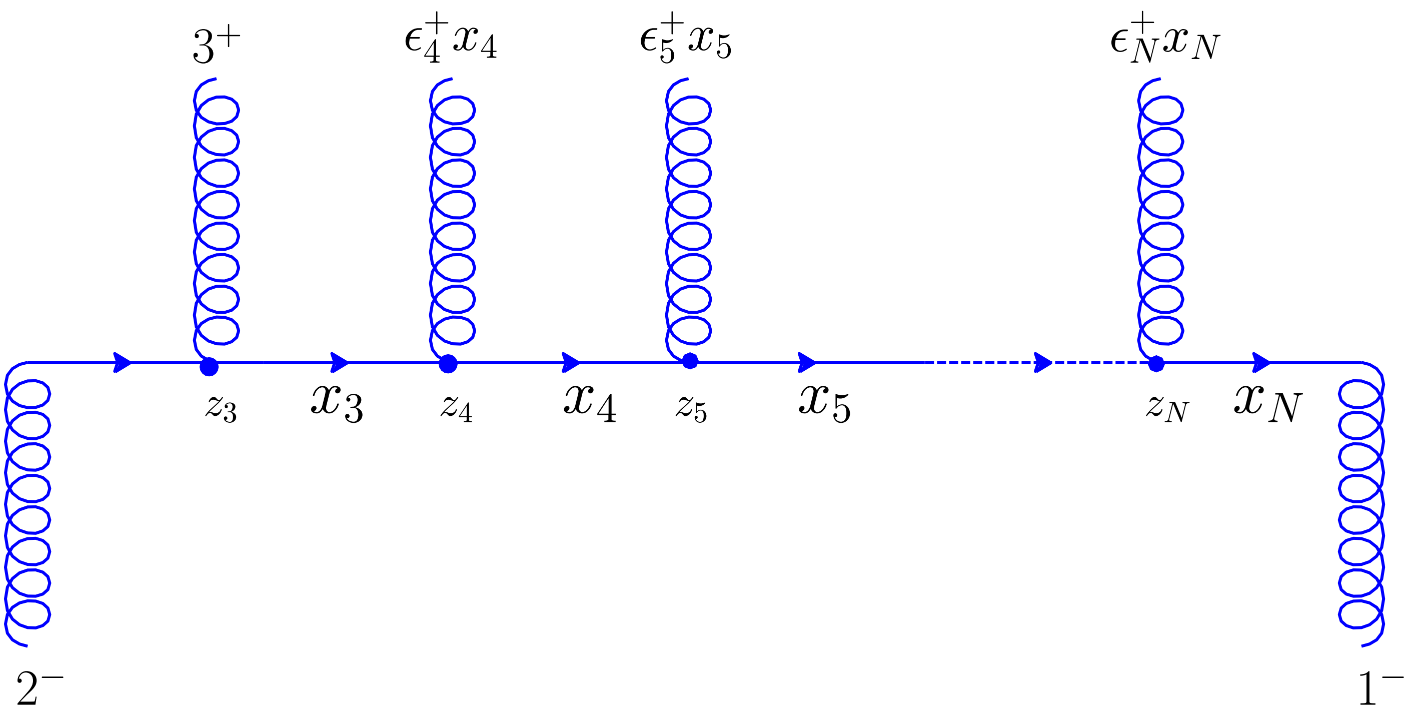
$$\times\langle 2|3|4_\sigma]\,\langle 2|3+4_\sigma|5_\sigma]\cdots\langle 2|3+4_\sigma+\ldots+(N-1)_\sigma|N_\sigma]$$

$$\uparrow$$

$$\langle 23\rangle[34_\sigma] = \langle 2|p_3|4_\sigma]$$

MHV Formula for Superstrings

in terms of dual conformal coordinates



$$x_n = \sum_{k=2}^{n} p_k$$

Conclusions

- Now we know for sure that scattering amplitudes, both in gauge theories and in superstring theory, are given by simple expressions. It is very tedious though to compute them by using traditional perturbative techniques like Feynman or string world-sheet diagrams …
- We can partially explain simplicity as a consequence of conformal and dual conformal symmetries, with twistors playing a prominent role.
- We need to find more shortcuts. Some shortcuts like Berends-Giele and BCFW recursion relations, and unitarity-based techniques already exist but they do not lead to a deeper understanding of QFT. S-matrix does not look anymore like a regular observable but it appears as the Matrix. One good shortcut can take us inside …